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RIGOROUS HIGH SPEED SEPARATION OF ZEROS
OF RIEMANN'S ZETA FUNCTION, III

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Rigorous high speed separation of zeros of Riemann's zeta function, III

by

J. van de Lune & H.J.J. te Riele

ABSTRACT

This is an intermediate report for the purpose of announcing that the first 300,000,001 zeros of the Riemann zeta function $\zeta(s)$ in the critical strip are simple and lie on the line $\operatorname{Re}(s) = \frac{1}{2}$. This extends our previous result that the first 250,000,000 zeros have this property.

KEY WORDS & PHRASES: *Riemann hypothesis, Riemann zeta function, Riemann-Siegel formula, Rosser's rule*

1. INTRODUCTION

This note is a continuation of our report NN 26/82 [2]. The purpose of the present note is to announce that *the first 300,000,001 zeros of the Riemann zeta function $\zeta(s)$ in the critical strip are simple and lie on the line $\text{Re}(s) = \frac{1}{2}$* . This result was obtained by extending the computations (up to zero # 250,000,000) described in [2]. Extending the computations a little further we found 4 Gram blocks in $[g_{300,000,000}, g_{300,000,004})$, all of them satisfying Rosser's rule. By applying Theorem 3.2 of BRENT [1] we completed the proof of our claim.

In Section 2 we present tables similar to those given in Section 2 of [2], this time for the range $[g_{249,999,999}, g_{300,000,000})$. The FORTRAN/COMPASS program by which the zeros in this range were separated is the same as that explicitly presented in [2].

We hope to extend our computations in the near future.

2. THE TABLES

In Table 2.1 we list the number of Gram blocks of type (L,k) , $1 \leq L \leq 8$, $1 \leq k \leq L$, in the interval $[g_{249,999,999}, g_{300,000,000})$. Similarly as in NN 26/82 these counts are *exact*.

On the lines with $L = 2$ and $L = 3$ in Table 2.1 we also list the number of exceptions to Rosser's rule of length 2 (with "0 0"-zero pattern) and those of length 3 (with "0 1 0"-zero pattern), respectively. Moreover, on the line with $L = 2$ we also mention the three blocks with "2 2"-zero pattern found in relation to the exceptions of type 5 and 6 (cf. Table 2.3). One exception to Rosser's rule of length $L = 3$ was found. Note the occurrence of two blocks of type $(7,1)$, not observed before, and one block of type $(7,7)$, which is the second one observed (the first one being $B_{195,610,937}$). The entries in parentheses are the approximate percentages with respect to the total number of blocks of length L , given in the final column.

For all Gram blocks our strategy of finding the "missing two zeros" was the same as in [2]. Table 2.1 shows the justification of this strategy.

Table 2.1

Number of Gram blocks of type (L, k) , $1 \leq L \leq 8$, $1 \leq k \leq L$, in the interval $[g_{249,999,999}, g_{300,000,000})$

L ↓	k →								Total
	1	2	3	4	5	6	7	8	
1	34,764,288								34,764,288
2	2,643,484 (50)	2,643,854 + 63 blocks with 0 0 zero-pattern (50) + 3 blocks with 2 2 zero-pattern							5,287,404
3	565,054 (47)	62,319 (5)	564,937 + 1 block with 0 1 0 zero-pattern (47)						1,192,311
4	107,657 (46)	9,992 (4)	9,979 (4)	107,578 (46)					235,206
5	10,511 (40)	2,159 (8)	1,123 (4)	2,195 (8)	10,388 (39)				26,376
6	284 (17)	442 (26)	137 (8)	153 (9)	448 (26)	256 (15)			1,720
7	2 ^{*)}	44	21	4	26	34	1 ^{*)}		132
8	0	0	1 ^{**)}	0	0	2 ^{**)}	0	0	3
^{*)} viz. B_n , for $n = 258,779,892$ 282,307,390 299,608,968 respectively.									
^{**)} viz. B_n , for $n = 264,680,348$ 258,666,950 262,831,140 respectively.									

In Table 2.2 we present the 64 exceptions to Rosser's rule in the range $[g_{249,999,999}, g_{300,000,000})$, including the local extreme values of $S(t)$. (The definitions of the various types are implicitly given in Table 2.3).

Table 2.2

*The exceptions to Rosser's rule of length 2 and 3
in the interval $[g_{249,999,999}, g_{300,000,000})$*

63 exceptions of length 2 with zero pattern "0 0".

Notation: n (type) local extreme $S(t)$,
where n is the index of the Gram block
 $B_n = [g_n, g_{n+2})$ containing no zeros

250076378(1)	-2.036397	272096379(2)	2.001554	284764536(2)	2.001422
252442158(2)	2.085094	272583009(1)	-2.032037	286172640(2)	2.042925
252904232(2)	2.112235	274190882(2)	2.008416	286688824(1)	-2.046407
255145220(1)	-2.002286	274268747(1)	-2.018420	287222173(2)	2.048065
255285972(2)	2.034861	275297430(2)	2.014738	287235535(2)	2.024894
256713230(1)	-2.015377	275545477(2)	2.032087	287304862(2)	2.003208
257992082(1)	-2.042307	275898480(2)	2.005068	287433571(1)	-2.021945
258447957(6)	2.005655	275953000(1)	-2.007296	287823551(1)	-2.038399
259298046(2)	2.091955	277117197(5)	-2.069283	287872423(2)	2.016959
262141503(1)	-2.006009	277447311(2)	2.058999	288766616(2)	2.024072
263681744(2)	2.006016	279059658(2)	2.037126	290122964(2)	2.039001
266617122(1)	-2.046423	279259145(2)	2.033129	290450849(5)	-2.068090
266628045(2)	2.048158	279513637(2)	2.000375	291426142(2)	2.075533
267305763(1)	-2.028836	279849070(2)	2.048163	292810354(2)	2.048278
267388405(2)	2.012716	280291419(1)	-2.021221	293109862(2)	2.013978
267441673(2)	2.085691	281449426(2)	2.000609	293398055(2)	2.042772
267464886(1)	-2.006418	281507954(2)	2.001841	294134427(2)	2.043302
267554908(2)	2.112706	281825600(1)	-2.033191	294216438(1)	-2.005490
269787480(1)	-2.080890	282547094(2)	2.002833	295367142(2)	2.049246
270881434(1)	-2.026487	283120964(2)	2.028096	297834112(2)	2.022351
270997584(2)	2.021752	283323493(1)	-2.032511	299099970(2)	2.030191

1 exception of length 3 with zero pattern "0 1 0".

$B_{266,527,881}$, followed by a Gram block of length 1 containing 3 zeros; local extreme $S(t) = -2.008550$.

Table 2.3 contains the frequencies of occurrence of the various types of exceptions to Rosser's rule in the range $[g_{249,999,999}, g_{300,000,000})$. Note that the number of exceptions of type 2 is twice the number of exceptions of type 1. There are neither exceptions of type 3 nor of type 4 in the range under consideration.

Table 2.3

Various types of exceptions to Rosser's rule and their frequencies in

$[g_{249,999,999}, g_{300,000,000})$

Exceptions of length 2

Gram block of length 2 without any zeros							type	frequency
g_{n-2}	g_{n-1}	g_n	g_{n+1}	g_{n+2}	g_{n+3}	g_{n+4}		
		0	0	3			1	20
	3	0	0				2	40
		0	0	4	0		3	0
0	4	0	0				4	0
		0	0	2	2		5	2
2	2	0	0				6	1

Exceptions of length 3

Gram block of length 3 containing 1 zero						frequency
g_{n-1}	g_n	g_{n+1}	g_{n+2}	g_{n+3}	g_{n+4}	
3	0	1	0			0
	0	1	0	3		1

Finally, from Tables 2.1 and 2.2 we have counted the following numbers of Gram intervals in $[g_{249,999,999}, g_{300,000,000})$ containing exactly m ($0 \leq m \leq 4$) zeros.

m	0	1	2	3	4
#	6,832,292	36,424,554	6,654,012	89,143 ^{*)}	0
%	13.7	72.8	13.3	0.2	0.0

4. REFERENCES

- [1] BRENT, R.P., *On the zeros of the Riemann zeta function in the critical strip*, Math. Comp., 33 (1979) pp. 1361-1372.
- [2] LUNE, J. VAN DE & H.J.J. TE RIELE, *Rigorous high speed separation of zeros of Riemann's zeta function*, II, Report NN 26/82, June 1982, Mathematical Centre, Amsterdam.

^{*)} We take the opportunity to correct a misprint in the corresponding table in [2]: the printed number 88,312 should read 88,314.